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A Study in Algebraic Graph Theory

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Abstract:

One of the important branches of Mathematics is Graph theory. In which algebraic methods applied to graph problems is known as algebraic graph theory. A study in algebraic graph theory is carried out in this article and a new concept of Algebraic and Geometric multiplicity to find the maximum matching of an undirected graph is introduced and discussed.

Key Words: Graph Theory, Matching, Maximum Matching, Geometric Multiplicity, complete graph, dense graph.

AMS Classification Key: 05C, 05C70, 911368, 15A18

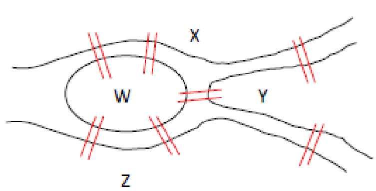
1. Introduction:

One of the emerging branches of Mathematics is graph theory. In Graph theory the study of algebraic graph theory and its application is one of the interesting area for researchers. Many research articles have been published and algebraic techniques is increasingly used in graph structures. In this study, the concept of algebraic graph theory and its applications are highlighted.

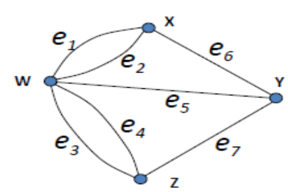
2. Graph Theory:

In mathematics, the study of mathematical structures is graph theory which is used to model pair wise relations between objects. A graph is made up of vertices (nodes) which are connected by edges (lines).

For example,



Konigsberg Bridge

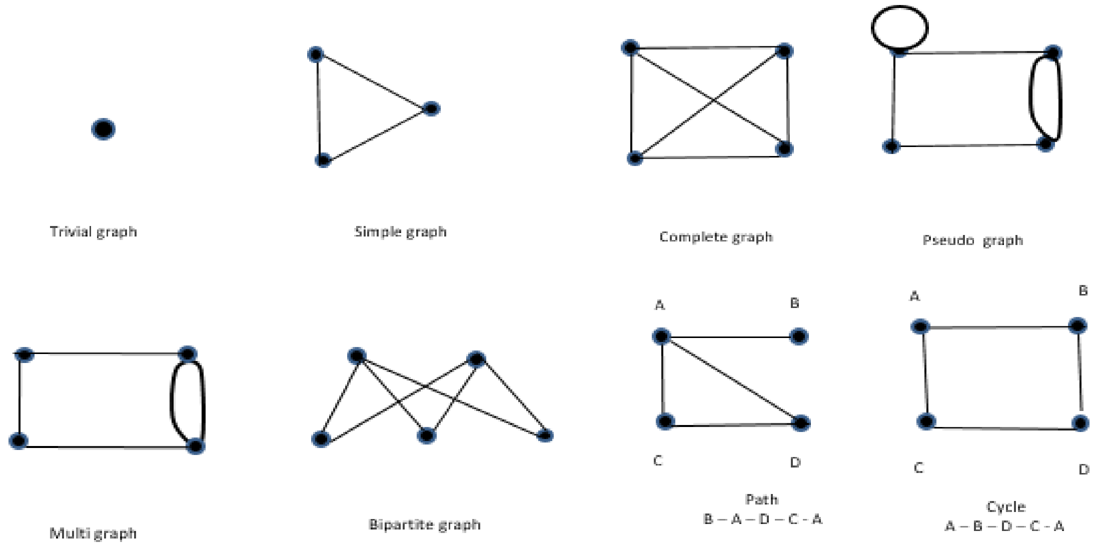


Graph

In the above figure the Konigsberg Bridge is converted to a graph, G= {V, E} where vertices V= {w, x, y, z} are the region on either side of the river and edges E = {e_1, e_2, e_3, e_4, e_5, e_6, e_7} are the path over the bridge.

The concept of graphs in Graph theory consists of some basic terms such as Degree of vertices, Adjacency matrix, Incidence matrix, Path, Cycle, Walk , etc., and it also has some basic different types of graphs.

For example,



3. Algebraic Graph theory:

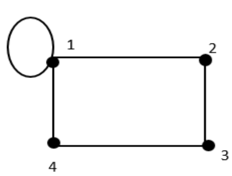
Algebraic tools are used for elegant proofs in algebraic theory and there are many interesting algebraic objects associated with graphs. Now there are more books dealing with various aspects of this subject**.** The books written by (N Biggs, 1993) and (Godsil and Royle, 2001) in algebraic graph theory contains massive information. Here we study the research carried in algebraic graph theory mainly Linear and Abstract algebra.

Some basic definitions and example are explained before studying in detail about algebraic graph theory.

An n x n matrix is said to be a symmetric matrix if the elements $a_{ij} = a_{ji}$ or the transpose of matrix is the matrix itself, and in graph theory the adjacency matrix of any undirected graph with or without self-loops is a symmetric matrix.

In graph theory the adjacency matrix is a square matrix of a finite graph whose components are expressed as 1 if the pair of vertices of the graph is adjacent otherwise it is zero.

For example,



$$A = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

Undirected graph with self-loop Symmetric and adjacency matrix

The Eigen values of the adjacency matrix of a graph is known as the spectrum of graph.

For example, consider a complete graph with 3 vertices and 3 edges

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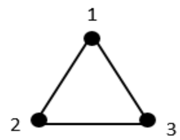


$$A=\begin{bmatrix}0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0\end{bmatrix}$$

The characteristic polynomial of the adjacency matrix is,

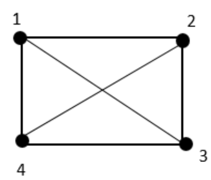
$$\lambda^3 - 3\lambda - 2 = 0$$

$\lambda = -1, -1, 2$ are the Eigen values of the graph and it's the spectrum of graph.

The diagonal elements of a diagonal matrix in a graph theory is the degree of its vertices and it is known as the degree matrix of the graph.

The Laplacian matrix in graph theory is formed by an undirected graph with no self-loops and multiple edges and it is given as L = D – A, D is the degree matrix and A is the adjacency matrix of the graph.

For example, consider an undirected graph with 4 vertices and 6 edges



$$A=\begin{bmatrix}0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 0\end{bmatrix} \qquad D=\begin{bmatrix}3 & 0 & 0 & 0\\ 0 & 3 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}$$

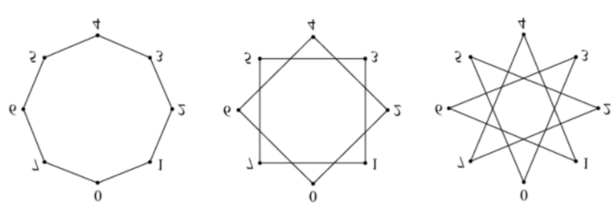
Adjacency matrix Degree matrix

The Laplacian matrix is given by,

$$L=D-A=\begin{bmatrix}3 & 0 & 0 & 0\\ 0 & 3 & 0 & 0\\ 0 & 0 & 3 & 0\\ 0 & 0 & 0 & 3\end{bmatrix}-\begin{bmatrix}0 & 1 & 1 & 1\\ 1 & 0 & 1 & 1\\ 1 & 1 & 0 & 1\\ 1 & 1 & 1 & 0\end{bmatrix}=\begin{bmatrix}3 & -1 & -1 & -1\\ -1 & 3 & -1 & -1\\ -1 & -1 & 3 & -1\\ -1 & -1 & -1 & 3\end{bmatrix}$$

The circulant graph in a graph theory is an undirected cyclic group of graph symmetries which takes any vertex to any other vertex.

For example from (Rajasingh, 2013),



Algebra of graph is giving an algebraic structure to an undirected graph in graph theory and it has many uses in the field of universal algebra.

For example,

Consider a graph G whose vertices are positive integers and it is represented by a pair (V,E) where V is the set of vertices and E subset of V x V is the set of edges. Then the algebraic structure is given by,

The structure $(G, +, \rightarrow, \varepsilon)$ introduced in the above satisfies the usual laws,

- ε is the empty set of the graph.
- $(G, +,\ \varepsilon)$ is an idempotent commutative monoid.
- $(G, \rightarrow, \varepsilon)$ is a monoid.
- → distributive over +, (e.g.) 1 → (2+3) = (1 → 2) + (1 → 3)

3.1. Study of Graph theory using Linear Algebraic concept:

In modern presentations of geometry the fundamental is linear algebra including for defining basic objects like lines, planes and rotations.

Algebraic graph theory in linear algebra is the study of the spectrum of adjacency matrix or the Laplacian matrix known as spectral graph theory is the first branch of algebraic graph theory which involves the study of graph in connection with linear algebra. A diagonalizable matrix can be factorized to a canonical form which represents the matrix in terms of Eigen values and Eigen vectors are known as its spectrum.

Spectral graph theory is the study of the properties of a graph in relation to its characteristic polynomial, Eigen values and Eigen vectors of matrices associated with the graph. A real symmetric matrix is orthogonally diagonalizable and its Eigen values are real algebraic integers.

The adjacency matrix of a finite simple graph is a (0-1) matrix with zeros on its diagonal and the matrix is symmetric if it is an undirected graph. In spectral graph theory the relation between a graph and the Eigen value, Eigen vectors of its adjacency matrix is studied.

Linear Algebra Applied to graph theory (Paul M Nguyen, 2017) is an application to linear algebra using graph theory and it is a best method of studying traffic flow for its networking, Electronic circuits or delivery routes. To solve this problem various matrix operations that are well –defined on the adjacency matrix is considered and established method for representing a graph mathematically.

Graph theory and Linear algebra (Dylan, 2017) explains the relationships between graph theory associated with matrix representations and the matrix properties found in linear algebra. It identifies certain unique properties of special classes of graphs such as complete graphs and acyclic graphs and how their specialty in graph theory reflects in their matrix properties.

Thermodynamic characterization of networks using graph polynomials (Cheng Ye, 2015) is a representation of graph structure based on a characteristic polynomial from the normalized Laplacian matrix show how the polynomial is linked to the Boltzmann partition function of network. This allows to find a number of thermodynamic quantities for the network, including the average energy and entropy. Also these thermodynamic variables can be computed in terms of simple network characteristics, e.g., the total number of nodes and node degree statistics for nodes connected by edges. The resulting Thermodynamic characterization can be applied to real-world time-varying networks representing complex systems in the financial and biological domains.

An algebraic Representation of Graphs and Applications to Graph Enumeration (Mestre, 2013) is a recursion formula to generate all the equivalence classes of connected Graphs with coefficients given by the inverses of the orders of their groups of Automorphisms. An algebraic graph representation is used to apply the result to the Enumeration of connected graphs, all of whose biconnected components have the same number of vertices and edges. The proof uses Abel's binomial theorem and generalizes Dziobek's induction proof of Cayley's formula.

Matrices of zeros and ones with fixed row and column sum vector (Richard, 1980) it consist set of all m x n matrices of 0's and 1's having ri1's row i and sj1's in column j has proved new results. The results can also be formulated in the set of bipartite graphs with bipartition into m and n vertices having degree sequence R and S respectively. Also it can be formulated as a set of hyper graphs with m vertices having degree sequence R and n edges whose cardinalities are given as S.

3.2. Study of Graph Theory in Abstract Algebra concept:

The study of algebraic structure is known as Group theory. Rings, Fields are well-known algebraic structures with some additional operations and axioms.

Algebraic graph theory in group theory specially involves study of Automorphisms groups and geometric group theory such as symmetric graphs, vertex –transitive, edge – transitive graphs etc. The study of group theory in algebraic graph theory is also related to the symmetry property of spectrum which is seen in its adjacency matrix.

The symmetry form of a graph which is mapped onto itself while preserving edge- vertex connectivity is known as Automorphism of a graph and it is defined for both directed and undirected graphs.

The study of finitely generated groups creating connection between algebraic properties of such groups and geometric properties in which the group acts. Also finitely generated groups itself act as geometric objects is an important idea of geometric graph theory.

Applications of Group Amalgams to algebraic graph theory (Ivanov, 1994) it deals with an example of sub group of Automorphism group and a graph. The group act as s-transitively on the graph if it act transitively on the set of paths of length s in the graph. This is proved as a theorem for 1-transitive group, s will be the largest integer such that subgroup acts s-transitively.

Some application of graph theory to finite groups (Edward A Bertram, 1982) has a theoretic result concerning the degree sequence, Vertex colorings and Vertex independence number which are used to derive theorems about finite groups.

Review and application of group theory to molecular systems biology (Edward A Rietman, 2011) it provides a mathematical idea to understand the boundary between living and non-living systems. In this group theory and abstract algebra is applied to molecular systems biology.

4. Algebraic and Geometric Multiplicity of and Undirected Graph:

Matching Theory is also one of the important branches of Graph theory in which the maximum cardinality of different types of graphs in both directed and undirected have been studied and many researchers have proved different theorems.

The study of finding maximum matching of an undirected graph through algebraic theory based on its multiplicity of Eigen values is a new concept. In algebraic theory the study of Linear Algebra involves the spectrum of its Adjacency is the first branch of Algebraic graph.

This concept was studied by Yunyun Yang and Gang Xie for the research article "Maximum Matching of a Digraph Based on the Largest Geometric Multiplicity" (Yunyun, 2016). The same concept was used to find maximum matching of an undirected graph based on Geometric and Algebraic Multiplicity.

Let A be an n × n matrix with Eigen value λ. Then the algebraic multiplicity of λ is the number of times λ is repeated as a root of the characteristic polynomial. The Geometric multiplicity does not require a new technique because it is the dimension of the null space.

The algebraic multiplicity and geometric multiplicity of Eigen values can differ but the geometric multiplicity never exceed algebraic multiplicity. Also if for every Eigen value of A, if the geometric multiplicity equals the algebraic multiplicity then A is said to be diagonalizable.

Application of algebraic multiplicity in population is studied in the article (Xue-zhi Li, 2001), in that article the algebraic multiplicity of complex Eigen values of population operator is discussed. It is proved that under condition the complex Eigen value of this operator is atmost algebraic multiplicity 1 and it results in an asymptotic expansion of the solution of corresponding population system.

4.1. Algebraic Multiplicity of an undirected complete graph:

If adjacency matrix of a complete graph with unit weight in the absence of Self- loops is of the form

$$\mathbf{A} = \begin{bmatrix} \mathbf{0} & \mathbf{1}\ldots & \mathbf{1} \\ \mathbf{1} & \mathbf{0}\ldots & \mathbf{1} \\ \mathbf{1} & \mathbf{1}\ldots & \mathbf{0} \end{bmatrix}$$

The characteristic polynomial of the adjacency matrix is,

$$|\lambda I_N - A| = [\lambda - (N-1)](\lambda+1)^{N-1}$$

Then the Eigen values and the corresponding Algebraic multiplicities are, if $\lambda_1 = N-1 \; then \; \delta(\lambda_1) = 1$ and if $\lambda_2 = -1 \; then \; \delta(\lambda_2) = N-1$

$$\therefore \sum_{j=1}^{l} \delta(\lambda_j) = \delta(\lambda_1) + \delta(\lambda_2) = 1 + N - 1 = N$$

Therefore, all the nodes are matched nodes.

4.2. Geometric Multiplicity of an undirected Sparse Graph and Dense graph (Yuan, 2013):

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The geometric multiplicity is the number of linearly independent eigenvectors of Eigen values. For an undirected graph the largest geometric multiplicity $\mu(\lambda_j)$ of the Eigen value $\lambda_j$of A_j

$$\mu(\lambda_j) = \dim V_{\lambda_j} = N - rank\,\{\lambda_j I_N - A\}$$

Where $\lambda_j (j = 1, 2, 3, \ldots, N)$represent the distinct Eigen values of A and I_N is the unit matrix with the same as A.

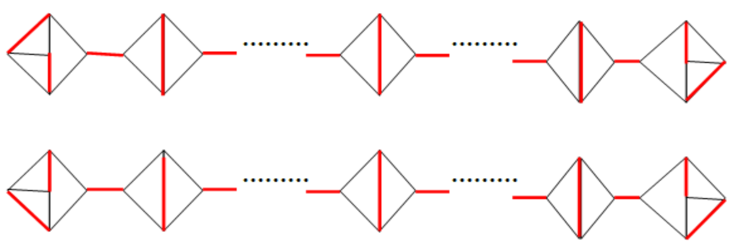
If an undirected graph is a sparse graph ((i.e.) the number of edges is much less than the possible number of edges) and if the rows of the adjacency matrix are linearly independent the Eigen value$\lambda_j = 0$ $(j = 1, 2, 3, \ldots, N)$. Then the Geometric Multiplicity $\mu(\lambda_j) = 0$. This gives that all the nodes of the graph is perfectly matched and the corresponding edges are perfectly matched. Since the maximum matching based on geometric multiplicity gives the number of unmatched nodes.

If an undirected graph is a Dense graph ((i.e.) the number of edges is near to the possible number of edges) then $\det(\lambda_m I_N - A) = 0$ if $\lambda = -1$ and it is the Eigen value of Largest Geometric Multiplicity.

Then column transformation is performed to the matrix $(\lambda_m I_N - A)$ to obtain column canonical and the linearly independent rows are combined with linearly dependent rows to form a new matrix which gives the matched nodes and the corresponding matched edges.

4.3. Perfect Matching of Bridge Graph based on Geometric and Algebraic Multiplicity (Jini.J, 2020)

A cubic undirected bridge graph is perfectly matched if n_i- Bridges (i=1, 2, 3… n) are connected in a single path. For an undirected cubic graph with zero bridge the perfect matching is based on Algebraic multiplicity and for n_i - Bridges the perfect matching is based on Geometric multiplicity. In this paper an undirected cubic graph connected in a single path has been found up to 3 bridges and the concept is extended to n_i - Bridges.



4.4. A new definition of geometric multiplicity of eigenvalues of tensors and some results based on it (Li, 2015)

A new definition of geometric multiplicity of eigenvalues of tensors, and based on this and the study of geometric and algebraic multiplicity of irreducible tensors' eigenvalues gives a result that the eigenvalues with modulus have the same geometric multiplicity. Also it has been proved that two-dimensional nonnegative tensors' geometric multiplicity of eigenvalues is equal to algebraic multiplicity of eigenvalues.

5. Conclusion:

In this paper, literature survey is made for the study of algebraic theory in Graph theory by the definitions and uses of first three branches of Algebraic theory. Also some research articles based on algebraic theory was studied to perform further research in the field of algebraic Graph theory. Finally my research in algebraic graph theory for finding maximum matching of an undirected graph based on algebraic and Geometric multiplicity is explained briefly.

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